16. KASSIR M.K. and SIH G.C., Mechanics of Fracture, 2, Three-dimensional Crack Problems. Noordhoff Intern. Publ., Leyden, 1975.
17. UFLYAND YA.S., The Method of Dual Equations in Problems of Mathematical Physics, Nauka, Leningrad, 1977.
18. BATEMAN H. and ERDELYI A., Higher Transcendental Functions, 2, /Russian translation/, Nauka, Moscow, 1974.
19. PRUDNIKOV A.P., BRYCHKOV YU.A. and MARICHEV O.I., Integrals and Series. Elementary Functions. Nauka, Moscow, 1981.
20. GUZ A.N., Stability of Elastic Bodies under Finite Strains. Naukova Dumka, Kiev, 1973.

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# ASYMPTOTIC SOLUTIONS OF INTEGRAL EQUATIONS OF CRACK THEORY PROBLEMS FOR THIN PLATES*

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Integral equations to which problems of the bending of thin plates with slits can be reduced are considered. On the basis of the properties of the integral equation kernels, conclusions are drawn concerning the classes of existence and uniqueness of their solutions. Asymptotic methods based on extraction of their principal part with subsequent exact inversion are proposed for the solution of the integral equations. On the basis of the solutions obtained, formulas are presented for the stress intensity factors in the slit angles, and their dependence on the geometrical parameters of the problem is shown. Other problems are indicated that result in the solution of the integral equations under consideration.

Asymptotic methods of solving integral problems of elasticity theory problems on cracks /1-3/ were considered earlier, as were also integral equations /4/ analogous to those considered below.

## 1. The integral equation.

Two kinds of problems (A and B) of crack theory for Kirchhoff-Love plates are studied.

*Problem A.* A Kirchhoff-Love plate in the form of a strip of width $2h\ (0 \leqslant y \leqslant 2h)$ is considered which is stiffly clamped along the edges. There is a rectilinear slit (crack) of length $2a$ on the plate axis of symmetry $(y = h)$. The slit (crack) edges are subjected to the action of a bending moment $M_y = \varphi_1(x)$. It is required to determine the angle of rotation of the slit edge $g_1^0(x)$ (Fig.1a).

*Problem B.* As in problem A, a plate in the form of a strip with a slit (crack) is considered. The slit (crack) is opened under the action of an antisymmetric transverse force $V_y = \varphi_2(x)$ distributed along the slit edges. Determine the vertical displacement of the slit (crack) edges $g_2^0(x)$ (Fig.1b).

The mathematical formulation of the problems under consideration is as follows: find the solution of the boundary value problem for the biharmonic equation

$$D\Delta^2 w = q(x, y) \tag{1.1}$$

($w(x, y)$ is the plate deflection, $q(x, y)$ is the distributed load, and $D$ is the cylindrical stiffness) with mixed boundary conditions.

*Problem A.*

$$\begin{gathered} w(x,0) = w_y'(x, 0) = V_y(x, h) = 0,\ |x| < \infty \\ M_y(x, h) = \varphi_1(x),\ |x| < a;\ w_y'(x, h) = 0,\ a < |x| < \infty \end{gathered} \tag{1.2}$$

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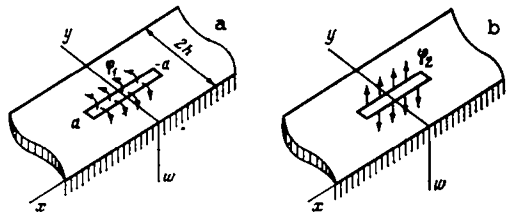


Fig.1

*Problem B.*

$$w(x,0) = w_y'(x,0) = M_y(x,h) = 0,\ |x| < \infty$$
$$V_y(x,h) = \varphi_2(x),\ |x| \leqslant a;\ w(x,h) = 0,\ a < |x| < \infty \tag{1.3}$$

By using a generalized Fourier integral transform, these boundary value problems can be reduced to the solution of the integral equation

$$\lambda^{-2m} \int_{-1}^{1} g_m(\xi)\, k_m\left(\frac{\xi - x}{\lambda}\right) d\xi = 2\pi \varphi_m(x), \quad |x| \leqslant 1 \tag{1.4}$$
$$\lambda = h/a,\ g_m(x) = (-1)^{m-1}(1-\nu)(3+\nu)\, D g_m^0/(2a)$$

Here (the integral is understood in the generalized sense)

$$k_m(t) = \int_{-\infty}^{\infty} K_m(u)\, e^{-iut}\, du \tag{1.5}$$
$$K_m(u) = \frac{2u}{(1-\nu)(3+\nu)} \frac{4\,\mathrm{ch}^2 u - (1+\nu)^2 \mathrm{sh}^2 u + (1+\nu)^2 u^2}{\mathrm{sh}\, 2u - (-1)^m 2u}$$

where $m = 1$ for problem *A* and $m = 2$ for problem *B*.

**2. Properties of the kernel of the integral equation.** The function $K_m(u)$ of the kernel of Eq.(1.4) is continuous along the real axis, is even, meromorphic in the complex plane and has the following asymptotic properties:

$$K_m = u^{2m-1} + O(e^{-2u}),\ |u| \to \infty;\ K_m = A_m + O(u^2),\ |u| \to 0 \tag{2.1}$$

In the complex plane $u = \sigma + i\tau$ the functions $K_m(u)$ can be represented in the form of infinite products

$$K_m(u) = K_m(0) \prod_{n=1}^{\infty} \frac{1 + u^2 \delta_{nm}^{-2}}{1 + u^2 \gamma_{nm}^{-2}} \tag{2.2}$$

where $\pm i\delta_{nm}$, $\pm i\gamma_{nm}$ are, respectively, the zeros and poles of the function $K_m(u)$.

*Lemma.* The representation

$$k_m(t) = \Gamma(2m)\, t^{-2m} - L_m(t), \quad L_m(t) = \int_0^{\infty} [u^{2m-1} - K(u)] \cos ut\, dt \tag{2.3}$$

is valid.

The functions $L_m(t)$ is regular in the strip $|t| < \infty$, $|\tau| < 2$. Moreover, for $|t| < 2$ it is represented by the absolutely convergent series

$$L_m(t) = \sum_{k=0}^{\infty} a_m^{(k)} t^{2k}, \quad a_m^{(k)} = \frac{(-1)^k}{(2K)!} \int_0^{\infty} [u^{2m-1} - K_m(u)]\, u^{2k}\, du \tag{2.4}$$

To prove the lemma we use the integral (in the generalized sense) /5/

$$\int_0^{\infty} u^{2m-1} \cos ut\, du = -\sin \pi \left(m - \frac{1}{2}\right) \Gamma(2m)\, t^{-2m}$$

where $\Gamma(z)$ is the Gamma function.

The function $L(z)$ is regular in the strip $|t| < \infty$, $|\tau| < 2$ for the estimate (2.1) and /6/. Representation (2.4) is obtained after expanding $\cos ut$ in a power series in t.

**3. Inversion of the principal part of the integral equation.** Extracting the singular part of the kernel of integral Eq.(1.4) by using (2.3), we obtain

$$\Gamma(2m)\int_{-1}^{1}\frac{g_m(\xi)\,d\xi}{(\xi-x)^{2m-1}} = -\pi\varphi_m(x) - \frac{1}{\lambda^{2m}}\int_{-1}^{1} g_m(\xi)\,L_m\left(\frac{\xi-x}{\lambda}\right)d\xi \tag{3.1}$$

Integrating the left and right sides of (3.1) $2m-1$ times with respect to $x$, we obtain the singular equation

$$\int_{-1}^{1}\frac{g_m(\xi)}{\xi-x}\,d\xi = \pi F_m(x) \tag{3.2}$$

$$\frac{d^{2m-1}F_m}{dx^{2m-1}} = -\varphi_m(x) - \frac{1}{\pi\lambda^{2m}}\int_{-1}^{1} g_m(\xi)\,L_m\left(\frac{\xi-x}{\lambda}\right)d\xi$$

whose solution can be obtained by reduction to a boundary value problem on the jump of analytic functions /7-9/. Seeking the solution of (3.2) in the class of functions $g_m(x) = G_m(x)(1-x^2)^{m-1/2}$ /10-12/, where $G_m(x) \in C_n[-1,1]$, we obtain the formula

$$g_m(x) = \pi\int_{-1}^{1} R_m(t,x)\,F_m(t)\,dt,\quad F_m(t) = \frac{1}{\pi^2(t-x)}\left(\frac{1-x^2}{1-t^2}\right)^{m-1/2} \tag{3.3}$$

in which the integral for $m = 2$ is understood as generalized in the sense of its finite part /13, 14/. The conditions

$$\int_{-1}^{1}\frac{t^{k-1}F_m(t)}{(1-t^2)^{m-1/2}}\,dt = 0,\quad k = 1,2,\ldots,(2m-1) \tag{3.4}$$

should be satisfied here.

*Theorem.* If the function $\Phi_n(x) \in B^{\alpha}_{n+1}[-1,1]$, $\alpha > m-1/2$, then any solution of integral Eq.(1.4) or (3.1) from the class $L_p[-1,1]$, $p > 1$ is a solution of the integral equation

$$g_m(x) = -(-1)^m\pi\int_{-1}^{1} R_m(t,x)\,\Phi_m(t)\,dt - \tag{3.5}$$

$$(-1)^m\lambda^{-2m}\int_{-1}^{1} R_m(t,x)\,dt\int_{-1}^{1} g_m(\xi)\,L_m^*((\xi-x)\lambda^{-1})\,d\xi,\ |x| \leqslant 1$$

of the form $g_m(x) = G_m(x)(1-x^2)^{m-1/2}$, where $G_m(x) \in C_n[-1,1]$ under the conditions (3.4) and (3.2) and vice-versa, Here

$$\frac{d^{2m-1}\Phi_m}{dx^{2m-1}} = \varphi_m(x),\quad L_m^*(t) = (-1)^m\lambda^{2m-1}\sum_{k=0}^{\infty}\frac{a_m^{(k)}}{b_m^{(k)}}\,t^{2k+2m-1}$$

$$b_1^{(k)} = 2k+1,\ b_2^{(k)} = (2k+1)(2k+2)(2k+3)$$

In the proof it is necessary to take into account that if $g(x) \in L_p[-1,1]$, $p > 1$, then the second component on the right-hand side of (3.1) is continuous with all derivatives of the function for $x \in [-1,1]$. Further, inverting the operator on the left-hand side of (3.1) by means of (3.3), we obtain (3.5).

**4. Large values of $\lambda$.** We use (3.5). We seek the solution of (1.4) in the form of a series in negative powers of $\lambda$

$$g_m(x) = \sum_{k=0}^{\infty} g_m^{(k)}(x)\,\lambda^{-2k} \tag{4.1}$$

Substituting (4.1) into (3.5) and equating the expressions obtained for identical powers of $\lambda$ we obtain a solution to $O(\lambda^{-6})$ accuracy in which

$$g_m^{(0)}(x) = -(-1)^m \pi \int_{-1}^{1} R_m(t,x)\,\Phi_m(t)\,dt \tag{4.2}$$

$$g_1^{(1)}(x) = -a_1^{(1)} \int_{-1}^{1} R_m(t,x)\,dt \int_{-1}^{1} (\xi - t)\, g_1^{(0)}(\xi)\,d\xi$$

$$g_1^{(2)}(x) = -\int_{-1}^{1} R_1(t,x)\,dt \int_{-1}^{1} (\xi - x)\left[\frac{a_1^{(1)}}{3}(\xi - t)\, g_1^{(0)}(\xi) + a_1^{(0)} g_1^{(1)}(\xi)\right] d\xi$$

$$g_2^{(1)}(x) = 0,\ g_2^{(2)} = -\frac{a_2^{(0)}}{6}\int_{-1}^{1} R_2(t,x)\,dt \int_{-1}^{1} (t-\xi)^2 g_2^{(k)}(\xi)\,d\xi$$

$$g_2^{(3)}(x) = -\frac{1}{60}\int_{-1}^{1} R_2(t,x)\,dt \int_{-1}^{1} [a_2^{(1)}(t-\xi)^5 g_2^{(0)}(\xi) + 10 a_2^{(0)}(t-\xi)^3 g_2^{(1)}(\xi)]\,d\xi$$

etc. In the important special case when $\varphi_m(x) = 1$, taking into account that $d^m\Phi/dx^m = \varphi_m(x)$, we obtain to accuracy $O(\lambda^{-2(m+2)})$

$$g_m(x) = \Gamma^{-1}(2m)\,(1-x^2)^{m-1/2}\left(1 + \sum_{k=1}^{m+1} \omega_m^{(k)}(x)\,\lambda^{-2k}\right) + O(\lambda^{-2(m+2)}) \tag{4.3}$$

$$\omega_1^{(1)} = \frac{a_m^{(0)}}{2} \qquad \omega_1^{(2)} = \frac{6\,(a_m^{(0)})^2 + (5+4x^2)\,a_1^{(1)}}{24}$$

$$\omega_2^{(1)} = 0, \quad \omega_2^{(2)} = \frac{a_2^{(0)}}{16}, \quad \omega_2^{(3)} = \frac{a_2^{(1)}\,(19+6x^2)}{960}$$

where $a_m^{(k)}$ is given by Eqs.(2.4).

5. Small values of $\lambda$. The zero-th term of the asymptotic form of the solution of the integral Eq.(1.4) can be constructed in the form /15, 16/

$$g_m(x) = \lambda^{2m-1} g_{m+}\left(\frac{1-x}{\lambda}\right) g_{m-}\left(\frac{1-x}{\lambda}\right) V_m^{-1}\left(\frac{x}{\lambda}\right) \tag{5.1}$$

under the condition that $V_m(x) \neq 0$ for $x \leqslant 1$. The functions $g_{m\pm}(x)$ satisfy the Wiener-Hopf integral equations and the function $V_m(x)$ satisfies the convolution equation

$$\int_0^\infty g_{m\pm}(\xi)\, k_m(\xi - x)\,d\xi = \pi\varphi_m(\lambda x \pm 1), \quad 0 \leqslant x < \infty \tag{5.2}$$

$$\int_{-\infty}^{\infty} V_m(\xi)\, k_m(\xi - x)\,d\xi = \pi\varphi_m(\lambda x), \quad |x| < \infty \tag{5.3}$$

The function $k_m(z)$ is given by (1.5).

We find the solution of integral Eqs.(5.2) and (5.3) for a special right-hand side $\varphi_m(\lambda x - 1) = e^{i\eta x}$ by using a generalized Fourier transform /5, 6/. To obtain a simpler form of the solution we approximate the function $K_m(u)$ of the kernel of the integral equation by an expression of two kinds

$$K_m(u) \approx \frac{r_m(u)}{\sqrt{u^2 + A_m^2}} H_m^{(N)}(u), \quad K_m(u) \approx \frac{u r_{m-1}(u)}{\operatorname{th} A_m u} H_m^{(N)}(u) \tag{5.4}$$

$$r_m(u) = \prod_{k=1}^{m} (u^2 + (a_m^{(k)})^2),$$

$$H_m^{(N)}(u) = \prod_{k=1}^{N} \frac{u^2 + (\delta_m^{(k)})^2}{u^2 + (\gamma_m^{(k)})^2}, \quad r_0(u) = H_m^{(0)}(u) = 1$$

that agree with $K_m(u)$ in the asymptotic properties (2.1). The constants $A_m$, $a_m^{(k)}$, $\delta_m^{(k)}$, $\gamma_m^{(k)}$ are found from the best approximation of $K_m(u)$ by these expressions on the real axis. To approximate $K_m(u)$ (1.5) with a 3% error along the real axis it is sufficient to take $N = 0$ in this case.

In the important special case when $\varphi_m(x) = 1$ ($\eta = 0$) the solution of (5.2) has the form

$(N = 0)$

$$g_{m\pm}(t) = \sqrt{K_m^{-1}(0)} \sum_{k=1}^{m+1} \chi\left(a_m^{(k)}, a_m^{(m+1-k)}, a_m^{(-k)}, t\right) \tag{5.5}$$

$$\chi(u, v, w, t) = \frac{\sqrt{A_m - u}}{(u - v)\, w} e^{-ut} \operatorname{erf} \sqrt{(A - u)\, t}$$

$$a_1^{(2)} = 1, \quad a_1^{(-1)} = a_1^{(-2)} = a_1^{(1)}, \quad a_1^{(0)} = -1, \quad a_1^{(2)} = 0$$

$$a_2^{(k)} = a_2^{(-k)}, \quad k = 1, 2; \quad a_2^{(3)} = 0, \quad a_2^{(0)} = a_2^{(1)}, a_2^{(-3)} = -a_2^{(2)}$$

for the approximation of the second form of (5.4) and

$$g_{m\pm}(t) = \frac{2}{\pi} \sqrt{A_m}\, z \sqrt{1 - z^2} \sum_{k=1}^{m} a_m^{(k)} F_1\left(1, 1 - \beta_m^{(k)}, \frac{3}{2}; 1 - z^2\right) \tag{5.6}$$

$$\beta_1^{(1)} = 0, \quad a_1^{(1)} = \sqrt{A_1}, \quad a_2^{(1)} = -a_2^{(2)} = \beta_2^{(1)} = a_{21}^{(1)}/\theta_2, \quad \beta_2^{(2)} = 0$$

$$z = \exp(-t/(2\theta_m)), \ \theta_m = A_m/\pi$$

for the approximation of the second form of (5.4). In particular, for $m = 1$ and $N = 0$ we obtain the simplest form of the solution

$$g_{1\pm}(t) = 2A_1\pi^{-1} \arcsin \sqrt{1 - \exp(-\pi t A_1^{-1})}$$

The solution of (5.5) is obtained easily by using a generalized Fourier transform

$$V(t) = K^{-1}(0) \tag{5.7}$$

Therefore, (5.1), (5.5), (5.6) and (5.7) yield a solution of integral Eq.(1.4) for small $\lambda$.

In combination the solutions for small and large $\lambda$ yield the solution of (1.4) in the whole range $\lambda \in (0, \infty)$. The nature of the solution obtained on approaching the edge is

$$g_m(x) = C(1 - x^2)^{m-1/2} \text{ as } |x| \to 1, \ C = \text{const} \tag{5.8}$$

**6. Numerical analysis of the solutions obtained.** An analysis performed on a computer shows that the solutions obtained for the integral equation for small and large $\lambda$ are identical for $\lambda \in [1, 2]$, as would be expected from theoretical considerations. The first form of the approximation (5.4) was used in analysing the solution. For $m = 1$ the solution for small $\lambda$ has the following values of the parameters: $A_1 = 2.4236$, $a_1^{(1)} = 1.4486$ while $a_0 = -0.5521$, $a_1 = 0.1911$ for solutions for large $\lambda$. In the case $m = 2$ the parameters are the following: $A_2 = 4.5$, $a_2^{(1)} = 1.6927$, $a_2^{(2)} = 2.020$ for small $\lambda$ and $a_0 = -10.012$, $a_1 = 26.1971$. For the junction, the error of the solutions does not exceed the error of the approximation, i.e., 3%.

**7. Stress intensity factor (SIF).** The solutions obtained enable us to present asymptotic formulas for the SIF. The SIF for problems A, B under consideration $K = K_1 - iK_2$ in the terminology of /10/ is calculated by means of the following formulas ($\delta$ is for large $\lambda$ and $\mu$ for small $\lambda$):

$$K_m^\delta = p_m g_m^\delta(\lambda)\, a^{m-1/2}, \quad K_m^\mu = p_m g_m^\mu\left(\frac{2}{\lambda}\right) h^{m-1/2}, \quad m p_m = 6\varphi_m^\circ h_0^2 \tag{7.1}$$

$$g_m^\delta(t) = \Gamma^{-1}(2m)\left[1 + \sum_{k=1}^{m+1} \omega_m^{(k)}(t)\, \lambda^{-2k}\right],$$

$$g_m^\mu(t) = \sum_{k=1}^{m+1} \chi\left(a_m^{(k)}, a_m^{(m+1-k)}, a_m^{(-k)}, t\right)$$

in which $\varphi_1^\circ$ is the moment distributed over the crack edge, $\varphi_2^\circ$ is the distributed transverse force, $h_0$ is the plate thickness, $\omega_m^{(k)}(x)$ is from (4.3) and $\chi(u, v, w, t)$ from (5.5). The asymptotic formulas (7.1) are identical for large and small $\lambda$ when $\lambda \in [1, 2]$ with up to 5% error. Note that the coefficient $K_m^\delta$ is proportional to $a^{m-1/2}$ and $K_m^\mu$ is proportional to $h^{m-1/2}$.

**8. Other problems about slits resulting in the integral Eq.(1.4).** We will mention two problems of the type B of practical importance, whose solution reduces to solving the integral Eq.(1.4).

*Problem B'*. A plate of the same planform with a slit of length $2a$ located on the axis of symmetry $(y = h)$ is considered. The slit opening is obtained by displacing the side face of the plane $(y = 2h, x < \infty)$ along the $Oz$ axis by an amount $2H$ relative to the side face $(y = 0)$. The side faces of the plate are here clamped stiffly. Determine the displacement of the slit edge $H - g_2'(x)$, $x < a$.

*Problem B''*. The problem of the opening of a slit of length $2a$ on the axis of symmetry

$(y = h)$ of a plate in the form of a strip $(0 \leqslant y \leqslant 2h, |x| < \infty)$ is considered. The plate side faces $(y = 0, y = 2h)$ are clamped and rotated through an angle $\beta$. Determine the displacement $g_2''(x)$ of the slit edge.

By using the generalized Fourier transform /13/ or the method of superposition of solutions of the homogeneous and inhomogeneous problems /3/, the problems can be reduced to the solution of integral Eq.(1.4). The solutions of the problems are here expressed in terms of the solutions $g_m(x)$ obtained for the integral equations (1.4) by the formulas $g_2'(x) = \gamma H K_2(0) h^{-3} \times g_2(x)$ (problem B') $g_2''(x) = \gamma\beta K_2(0) h^{-2} g_2(x)$ (problem B"), where $\gamma = 2(1-\nu)^{-1}(3+\nu)^{-1}D^{-1}$.

Note that the coefficients of problems B', B" are proportional to the following geometrical parameters $K_2'^{\delta} \sim a^{3/2}h^{-3}H$, $K_2'^{\mu} \sim h^{-3/2}H$, $K_2''^{\delta} \sim a^{3/2}h^{-2}$, $K_2''^{\mu} \sim h^{-1/2}$.

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REFERENCES

1. ALEKSANDROV V.M., On the theory of equilibrium cracks in an elastic layer. Stress Concentration /Russian translation/, 1, Naukova Dumka, Kiev, 1965.
2. SMETANIN B.I., Two slots in a band of finite width, PMM, 34, 2, 1970.
3. ALEKSANDROV V.M. and SMETANIN B.I., On longitudinal equilibrium cracks in plates. Proceedings of the Sixth All-Union Conf. on Plate and Shell Theory, Nauka, Moscow, 1966.
4. ONISHCHUK O.V.,POPOV G.YA. and FARSHAIT P.G., On the features of contact forces for the bending of a plate with thin inclusions, PMM, 50, 2, 1986.
5. BRYCHKOV YU.A. and PRUDNIKOV A.P., Integral Transforms of Generalized Functions. Nauka, Moscow, 1977.
6. NOBLE B., Application of the Wiener-Hopf Method to Solve Partial Differential Eqations, /Russian translation/, IIL, Moscow, 1962.
7. ZABREIKO P.P., KOSHELEV A.I., KRASNOSEL'SKII M.A., et al., Integral Equations. Nauka, Moscow, 1968.
8. GAKHOV F.D., Boundary Value problems. Nauka, Moscow, 1977.
9. MUSHELISHVILI N.I., Certain Fundamental Problems of the Mathematical Theory of Elasticity, Nauka, Moscow, 1966.
10. CHEREPANOV G.P., Mechanics of Brittle Fracture. Nauka, Moscow, 1974.
11. SIH G.C., PARIS P.C. and ERDOGAN F., Crack-tip stress-intensity factors for plane extension and plate bending problems. Trans, ASME, Ser. E., J. Appl. Mech. 29, 2, 1962.
12. LIN'KOV A.M. and MERKULOV V.A., Problems on the bending of plates with slits, Izv. Akad. Nauk SSSR, Mekhan. Tverd, Tela, 1, 1975.
13. VILENKIN N.YA., GORIN E.A., KOSTYUCHENKO A.G., et al., Functional Anaysis. Nauka, Moscow, 1964.
14. ONISHCHUK O.V. and POPOV P.YA., On certain problems of the bending of plates with cracks and thin inclusions, Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, 4, 1980.
15. VOROVICH I.I., ALEKSANDROV V.M. and BABESHKO V.A., Non-classical Mixed Problems of the Theory of Elasticity. Nauka, Moscow, 1974.
16. ALEKSANDROV V.M. and BABESHKO V.A., Contact problems for an elastic strip of small thickness, Izv. Akad. Nauk SSSR, Mekhanika, 2, 1965.

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